

Jensen, Hermite-Hadamard, and Ostrowski type inequalities for a generalized Class of convex stochastic functions

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Abstract

In this paper, we explore a set of generalized concepts associated with convex stochastic processes and their structural properties. Stochastic processes are among the most fundamental components of probability theory, providing a mathematical framework for modeling systems that evolve randomly over time. Recognizing the importance of convexity in analyzing such processes, we extend the classical notion of convexity and introduce the concept of a generalized modified pH -convex stochastic process. This newly developed framework enables a broader class of stochastic functions to be examined in various integral and inequality settings. Furthermore, we investigate the behaviour of these generalized processes with respect to well-established inequalities, specifically, those of Hermite, Hadamard, Jensen, and Ostrowski. By establishing these generalized inequalities, our results not only unify several existing findings but also offer new analytical tools that may lead to deeper insights in stochastic analysis, optimization, and applied probability.

1 INTRODUCTION

When there are probabilistic quantities in the literature, the convexity of stochastic processes is important in optimization, primarily in optimum designs. It is also beneficial for numerical approximations. A stochastic process is a mathematical tool used in probability theory and other related disciplines that are defined as a set of random variables. The random variables were verified to be connected to or listed by a large number of numbers, which were typically seen as emphases in time, resulting in the translation of a stochastic process, speaking to numerical estimations of some system that was randomly changing over time. For example, the growth of a bacterial population, the fluctuation of an electrical flow due to thermal noise, or the growth of a gas molecule. Stochastic processes are often used as scientific models of systems that appear to alter at random. They may be used in a variety of fields, including science, for example, Biology, chemistry, ecology, neurology, and physics, as well as engineering and technology fields. For example, signal processing, data theory, computer science, cryptography, and telecommunications are some of the topics covered in this course. Furthermore, seemingly random fluctuations in money-related markets have prompted the widespread use

of stochastic processes in fund management. For a comprehensive examination of convex functions, inequality theory, and applications, we refer [16][17][11][14][13] and references therein. In 1980 Nikodem[22] presented the notion of convex stochastic processes and certain characteristics that are also seen in traditional convex functions. Some types of convex stochastic processes were defined by Skowronski[26] in 1992.

Kotrys discovered the Hermite-Hadamard inequality for convex stochastic processes in 2012[15]. Much researches on the above-mentioned processes have been conducted in recent years. In the literature, there were several definitions of different convexity and some novel inequalities for these stochastic processes. The main and interesting work on stochastic process are[19][24][28][23][10].

The primary goal of this paper is to introduce the concept of a generalized modified(p,h)-convex stochastic process and to develop Hermite–Hadamard, Jensen, and Ostrowski type inequality. The main motivation for this paper is also to deliver the idea of generalized modified(p,h)-convex stochastic process.

The η -convex stochastic processes and modified (p,h)-convex stochastic processes will be discussed first, and then we will apply our concept to some fundamental and significant findings.

Before that, we'll go over several basic concepts, such as convex sets and convex functions. Some researches on generalization of convexity may be found here. [27][21][9][18][2][12][20][29][4].

2 Preliminaries

Here we start the concept of the stochastic process for a probability space. Consider space under probability space (Π, L, M) . A random variable is a function c , if ψ is A -measurable, whereas a stochastic process is a function $\psi : J \times \Pi \rightarrow \mathbb{R}$ if $\psi(s, \cdot)$ is a random variable, where $s \in J$.

2.1 Stochastic Process

2.1.1 Properties of Stochastic Process

- **Continuous**

The mapping $\psi : J \times \Pi \rightarrow \mathbb{R}$ is continuous on J , if $\forall v_o \in J$,

$$P - \lim_{v \rightarrow v_o} \psi(v, \cdot) = \psi(v_o, \cdot),$$

whereas $P - \lim$ represent the limit in probability.

- **Mean square continuous**

The stochastic process $\psi : J \times \Pi \rightarrow \mathbb{R}$ is mean square continuous on J , if for all $v_o \in J$,

$$\lim_{v \rightarrow v_o} \mathbb{D}(\psi(v, \cdot) - \psi(v_o, \cdot))^2 = 0,$$

whereas $\mathbb{D}|\psi(v, \cdot)|$ represent an expectation of random variable.

- **Mean square differentiable**

The stochastic process $\psi : J \times \Pi \rightarrow \mathbb{R}$ is differentiable at $s \in J$, if there is a random variable $\psi'(v, \cdot) : J \times \Pi \rightarrow \mathbb{R}$ such that $v_o \in J$,

$$\psi'(v_o, \cdot) = P - \lim_{v \rightarrow v_o} \left(\frac{\psi(v, \cdot) - \psi(v_o, \cdot)}{v - v_o} \right).$$

- **Mean square integral**

Consider the stochastic process $\psi : J \times \Pi \rightarrow \mathbb{R}$ with $\mathbb{D}[\psi(s, \cdot)] < \infty$.

We can say a random variable $w : \Pi \rightarrow \mathbb{R}$ to be mean square integrable of the process ψ on $[c_1, c_2]$, if for each normal sequence of partition of $[c_1, c_2]$, $c_1 = v_o < v_1, \dots, v_r = c_2$ for all $M_k \in [v_{k-1}, v_k]$ we have,

$$\lim_{n \rightarrow \infty} \mathbb{D} \left[\left(\sum_{k=1}^r \psi(M_k, \cdot)(v_k - v_{k-1}) - v(\cdot) \right)^2 \right] = 0,$$

$$v(\cdot) = \int_{c_1}^{c_2} \psi(s, \cdot) ds(a.e). \quad (2.1)$$

The monotonicity of mean square integral will be used mostly in this paper. If $\psi_1(v, \cdot) \leq \psi_2(v, \cdot)$ (a.e.) for the interval $[c_1, c_2]$, then

$$\int_{c_1}^{c_2} \psi_1(v, \cdot) dt \leq \int_{c_1}^{c_2} \psi_2(v, \cdot) dt. \quad (2.2)$$

The inequality (1.2) follows directly from the definition of the mean square integral.

Lemma 2.1. *If $\psi : J \times \Pi \rightarrow \mathbb{R}$ is a stochastic process of the form $\psi(s, \cdot) = A_1(\cdot)s + A_2(\cdot)$, where $A_1, A_2 : \Pi \rightarrow \mathbb{R}$ are random variable such that $E[A_1^2] < \infty$ and $[c_1, c_2] \subset J$, then*

$$\int_{c_1}^{c_2} \psi(s, \cdot) ds = A_1(\cdot) \frac{c_2^2 - c_1^2}{2} + A_2(\cdot)(c_2 - c_1). \quad (2.3)$$

Now, come to the definition of η -convex stochastic process.

2.2 η -convex stochastic process

Let (Π, A, P) be the probability space and $J \subset \mathbb{R}$ be an interval, then $\psi : J \times \Pi \rightarrow \mathbb{R}$ is an η -convex stochastic process, if

$$\psi(sm + (1 - s)n, \cdot) \leq \psi(n, \cdot) + s\eta(\psi(m, \cdot), \psi(n, \cdot)), \quad (2.4)$$

$\forall m, n \in J$, and $s \in [0, 1]$.

1 If we take $\eta(m, n) = m - n$, we obtain convex stochastic process. By taking $\psi(m, \cdot) = \psi(n, \cdot)$, we get

$$s\eta(\psi(m, \cdot), \psi(n, \cdot)) \geq 0,$$

for any $m \in J$, and $s \in [0, 1]$, which implies that

$$\eta(\psi(m, \cdot), \psi(n, \cdot)) \geq 0,$$

for any $m \in J$.

2 Also if we take $s = 1$, we get

$$\psi(m, \cdot) - \psi(n, \cdot) \leq \eta(\psi(m, \cdot), \psi(n, \cdot))$$

For any $m, n \in J$.

Because the first condition entails the second, if we try to define η -convex stochastic process on an interval J of a real numbers, we should assume that,

$$\eta(m, n) \geq m - n, \quad (2.5)$$

for any $x, y \in J$. It can be noticed that, if $\psi : J \rightarrow \mathbb{R}$ is stochastic process and $\eta : \psi(J) \times \psi(J) \rightarrow \mathbb{R}$ is an arbitrary bi-function that satisfies the above equation, then for any $m, n \in J$ and $s \in [0, 1]$, we have

$$\begin{aligned} \psi(sm + (1 - s)n, \cdot) &\leq \psi(n, \cdot) + s(\psi(m, \cdot) - \psi(n, \cdot)) \\ &\leq \psi(n, \cdot) + s\eta(\psi(m, \cdot), \psi(n, \cdot)), \end{aligned}$$

which tells that ψ is η convex stochastic process.

2.2.1 h-convex stochastic process

If a non negative $\psi : J \times \Pi \rightarrow \mathbb{R}$ be the h-convex stochastic process, if

$$\psi[(sm + (1 - s)n), \cdot] \leq h(s)\psi(m, \cdot) + h(1 - s)\psi(n, \cdot),$$

$s \in (0, 1)$ and $\forall m, n \in J$.

2.2.2 p-convex stochastic process

If a non negative $\psi : J \times \Pi \rightarrow \mathbb{R}$ be the p-convex stochastic process, if

$$\psi([sm^p + (1-s)n^p], \cdot)^{\frac{1}{p}} \leq (s)\psi(m, \cdot) + (1-s)\psi(n, \cdot),$$

$s \in (0, 1)$ and $\forall m, n \in J$.

2.3 Generalization of convex stochastic process

2.3.1 Generalized p-convex stochastic process

If a non negative $\psi : J \times \Pi \rightarrow \mathbb{R}$ be the generalized p-convex stochastic process, if

$$\psi([sx^p + (1-s)y^p], \cdot)^{\frac{1}{p}} \leq \psi(y, \cdot) + s\eta(\psi(x, \cdot), \psi(y, \cdot)),$$

$s \in [0, 1]$, and $\forall x, y \in J$.

2.3.2 (p,h)-convex stochastic process

Assume that $\psi, h : j \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative and non-zero function. A function $\psi : J \times \Pi \rightarrow \mathbb{R}$ where J is p-convex set in \mathbb{R} is called modified $(p, h) - convex$ stochastic process, if ψ is non-negative, and

$$\psi([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot) \leq h(s)\psi(m, \cdot) + h(1-s)\psi(n, \cdot),$$

for $s \in (0, 1)$ and $\forall m, n \in J$.

2.3.3 Modified (p,h)-convex stochastic process

Assume that $\psi, h : l \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative and non-zero function. A function $\psi : I \times \Pi \rightarrow \mathbb{R}$ where J is p-convex set in \mathbb{R} is called modified $(p, h) - convex$ stochastic process, if ψ is non-negative, and

$$\psi([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot) \leq h(s)\psi(m, \cdot) + (1-h(s))\psi(n, \cdot)$$

for $s \in (0, 1)$ and $\forall m, n \in J$.

2.3.4 Generalized modified (p,h)-convex stochastic process

Assume that $\psi, h : l \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative and non-zero function. A function $\psi : J \times \Pi \rightarrow \mathbb{R}$ where J is p-convex set in \mathbb{R} is called generalized modified $(p, h) - convex$ stochastic process, if ψ is non-negative and

$$(\psi[sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot) \leq \psi(m, \cdot) + \eta[h(s)(\psi(m, \cdot), \psi(n, \cdot))],$$

for $s \in (0, 1)$ and $\forall m, n \in J$.

3 Some basic results of generalized modified (p,h) convex stochastic process.

3.1 Proposition

Consider two modified (p, h) convex stochastic functions $\psi_1, \psi_2 : I \times \Pi \rightarrow \mathbb{R}$ such that

1. If two generalized modified (p, h) convex stochastic function ψ_1 and ψ_2 are additive then $\psi_1 + \psi_2 : I \times \Pi \rightarrow \mathbb{R}$ is generalized modified (p, h) convex stochastic process, where η is also generalized modified convex stochastic process.
2. If a generalized modified (p, h) function, ψ_1 is non-negatively homogeneous, then for any $\gamma \geq 0, \gamma\psi_1 : I \times \Pi \rightarrow \mathbb{R}$ is a generalized modified (p, h) convex stochastic process, where η is also generalized modified convex stochastic process.

Proof. If $\psi_1 : J \times \Pi \rightarrow \mathbb{R}$ and $\psi_2 : I \times \Pi \rightarrow \mathbb{R}$ then the generalized modified (p, h) convex stochastic process is,

$$\begin{aligned}
 & (\psi_1 + \psi_2) \left([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot \right) \\
 & \leq h(s) [(\psi_1 + \psi_2)(m, \cdot)] + (1-h(s))(\psi_1 + \psi_2)(n, \cdot) \\
 & \leq h(s)\psi_1(m, \cdot) + h(s)\psi_2(m, \cdot) + ((1-h(s))(\psi_1(n, \cdot) + \psi_2(n, \cdot))) \\
 & \leq h(s)\psi_1(m, \cdot) + h(s)\psi_2(m, \cdot) + (1-h(s))\psi_1(n, \cdot) + (1-h(s))\psi_2(n, \cdot) \\
 & \leq [h(s)\psi_1(m, \cdot) + (1-h(s))\psi_1(n, \cdot)] + [h(s)\psi_2(m, \cdot) + (1-h(s))\psi_2(n, \cdot)] \\
 & = \psi_1 \left([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot \right) + \psi_2 \left([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot \right).
 \end{aligned}$$

□

Proof. The proof of this proposition is straight forward. □

3.2 Proposition

If $\psi : [a, b] \rightarrow \mathbb{R}$ is generalized modified (p,h) convex stochastic functions, then

$$\max \psi(x, \cdot)_{x \in [b_1, b_2]} \leq \max \{ \psi(b_2, \cdot), \psi(b_2, \cdot) + \eta h(s)(\psi(b_1, \cdot), \psi(b_2, \cdot)) \}.$$

Proof. Consider,

$$x^p = sb_1^p + (1-s)b_2^p$$

for an arbitrary $x \in [b_1, b_2]$ and some $s \in [0, 1]$. We can write

$$\psi(x, \cdot) = \psi[(sb_1^p + (1-s)b_2^p)^{\frac{1}{p}}, \cdot].$$

since, ψ is an generalized modified (p,h) convex stochastic process, so by definition

$$\psi(x, \cdot) \leq h(s)\psi(b_1, \cdot) + (1-h(s))\psi(b_2, \cdot).$$

This can be written as,

$$\begin{aligned}
\psi(x, \cdot) &\leq h(s)\psi(b_1, \cdot) + \xi(b_2, \cdot) - h(s)\psi(b_2, \cdot) \\
\psi(x, \cdot) &\leq \psi(b_2, \cdot) + h(s)\psi(b_1, \cdot) - h(s)\psi(b_2, \cdot) \\
\psi(x, \cdot) &\leq \psi(b_2, \cdot) + h(s)(\psi(b_1, \cdot) - \psi(b_2, \cdot)). \\
\psi(x, \cdot) &\leq \psi(b_2, \cdot) + \eta h(s)(\psi(b_1, \cdot), \psi(b_2, \cdot))
\end{aligned} \tag{3.1}$$

and,

$$\psi(b_2, \cdot) + \eta h(s)(\psi(b_1, \cdot), \psi(b_2, \cdot)) \leq \max\{\psi(b_2, \cdot), \psi(b_2, \cdot) + \eta h(s)(\psi(b_1, \cdot), \psi(b_2, \cdot))\}. \tag{3.2}$$

since, x is arbitrary, so we can get our desired result. \square

3.3 Proposition

Assume $f_i : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be the generalized modified (p,h) convex stochastic function, suppose μ_1, \dots, μ_n be positive scalars. consider a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that ψ is generalized modified (p,h)-convex stochastic function.

Proof. we know that $f_i : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be the generalized modified (p,h) convex stochastic function. Then $s \in [0, 1]$ and $\forall x, y \in I$, we have

$$\begin{aligned}
\psi\left(sx^p + (1-s)y^p\right)^{\frac{1}{p}}, \cdot &= \sum_{i=1}^n \mu_i f_i\left(\left(sx^p + (1-s)y^p\right)^{\frac{1}{p}}, \cdot\right) \\
&\leq \sum_{i=1}^n \mu_i (h(s)f_i(x, \cdot) + (1-h(s))f_i(y, \cdot)) \\
&\leq \sum_{i=1}^n \mu_i (h(s)f_i(x, \cdot) + f_i(y, \cdot) - h(s)f_i(y, \cdot)) \\
&\leq \sum_{i=1}^n \mu_i (h(s)f_i(x, \cdot) - h(s)f_i(y, \cdot) + f_i(y, \cdot)) \\
&\leq \sum_{i=1}^n \mu_i (f_i(y, \cdot) + h(s)(f_i(x, \cdot) - f_i(y, \cdot))) \\
&= \sum_{i=1}^n \mu_i f_i(y, \cdot) + h(s) \left(\sum_{i=1}^n \mu_i (f_i(x, \cdot) - f_i(y, \cdot)) \right).
\end{aligned}$$

since,

$$\sum_{i=1}^n \mu_i f_i(x_i, \cdot) = \psi(x, \cdot)$$

and,

$$\sum_{i=1}^n \mu_i f_i(y_i, \cdot) = \psi(y, \cdot)$$

so,

$$= \psi(y, \cdot) + h(s)(\psi(x, \cdot) - \psi(y, \cdot))$$

$$\psi\left(sx^p + (1-s)y^p\right)^{\frac{1}{p}}, \cdot \leq \psi(y, \cdot) + h(s)\eta[\psi(x, \cdot), \psi(y, \cdot)].$$

Then the proof is complete. □

3.4 Proposition

Let $h : J \subset \mathbb{R} \rightarrow [0, 1]$. If $g : I \rightarrow \mathbb{R}$ is generalized modified (p,h)-convex stochastic function and $f : I \rightarrow \mathbb{R}$ is convex and increasing, then $f \circ g$ is also generalized modified (p,h)-convex stochastic function.

Proof. since, g is generalized modified (p,h)-convex stochastic function on I . We obtained,

$$\begin{aligned} f \circ g([su^p + (1-s)v^p]^{\frac{1}{p}}, \cdot) &= f\left(g([su^p + (1-s)v^p]^{\frac{1}{p}})\right) \\ &\leq f(h(s)g(u, \cdot) + (1-h(s))g(v, \cdot)). \end{aligned}$$

then by using the convexity of f , we obtained,

$$\begin{aligned} f(h(s)g(u, \cdot) + (1-h(s))g(v, \cdot)) &\leq h(s)f(g(u, \cdot)) + (1-h(s))f(g(v, \cdot)) \\ &= h(s)(f \circ g)(u, \cdot) + (1-h(s))(f \circ g)(v, \cdot) \\ &\leq h(s)(f \circ g)(u, \cdot) + (f \circ g)(v, \cdot) - h(s)(f \circ g)(v, \cdot) \\ &\leq (f \circ g)(v, \cdot) + h(s)(f \circ g)(u, \cdot) - h(s)(f \circ g)(v, \cdot) \end{aligned}$$

$$f \circ g([su^p + (1-s)v^p]^{\frac{1}{p}}, \cdot) \leq (f \circ g)(v, \cdot) + h(s)\eta((f \circ g)(u, \cdot), (f \circ g)(v, \cdot)).$$

□

3.5 Proposition

Let $h : J \subset \mathbb{R} \rightarrow [0, 1]$. Further, let $f_j : I \rightarrow \mathbb{R}, j \in \mathbb{N}$ is non empty collection of generalized modified (p,h)-convex stochastic process such that, for each $u \in I, \max_{j \in J} f_j(u, \cdot)$ exists in \mathbb{R} , then the function $f : I \rightarrow \mathbb{R}$ defined by $f(u, \cdot) = \max_{j \in J} f_j(u, \cdot)$, for each $u \in I$ is generalized modified (p,h)-convex stochastic process.

Proof. For any $u, v \in I$ and $s \in [0, 1]$, we have

$$\begin{aligned} f\left(su^p + (1-s)v^p\right)^{\frac{1}{p}}, \cdot &= \max_{j \in J} f_j\left([su^p + (1-s)v^p]^{\frac{1}{p}}, \cdot\right) \\ &\leq \max_{j \in J} \{h(s)f_j(u, \cdot) + (1-h(s))f_j(v, \cdot)\} \\ &\leq h(s)\max_{j \in J} f_j(u, \cdot) + (1-h(s))\max_{j \in J} f_j(v, \cdot) \\ &= h(s)f(u, \cdot) + (1-h(s))f(v, \cdot). \end{aligned}$$

by definition of stochastic,

$$\begin{aligned} h(s)f(u, \cdot) + (1-h(t))f(v, \cdot) &\leq h(s)f(u, \cdot) + f(v, \cdot) - h(s)f(v, \cdot) \\ &\leq f(v, \cdot) + h(s)f(u, \cdot) - h(s)f(v, \cdot) \\ &\leq f(v, \cdot) + h(s)[f(u, \cdot) - f(v, \cdot)] \\ &\leq f(v, \cdot) + h(s)\eta[f(u, \cdot), f(v, \cdot)] \end{aligned}$$

so,

$$f\left(su^p + (1-s)v^p\right)^{\frac{1}{p}}, \cdot \leq f(v, \cdot) + h(s)\eta[f(u, \cdot), f(v, \cdot)].$$

our required result is proved. □

3.6 Theorem

A random variable $\psi : J \times \Pi \rightarrow \mathbb{R}$ is generalized modified (p,h) convex stochastic process if and only if, for any $m_1, m_2, m_3 \in J$ with $m_1 \leq m_2 \leq m_3$, we have

$$\det \begin{vmatrix} h(m_3^p - m_2^p) & \psi(m_2, \cdot) - \psi(m_3, \cdot) \\ h(m_3^p - m_1^p) & \eta(\psi(m_1, \cdot), \psi(m_3, \cdot)) \end{vmatrix} \leq 0.$$

Proof. Suppose that ψ is generalised modified (p, h) convex stochastic process and $m_1, m_2, m_3 \in I$ such that, $m_1 \leq m_2 \leq m_3$. Then there exist $\beta \in (0, 1)$, such that

$$m_2^p = \beta m_1^p + (1-\beta)m_3^p$$

where,

$$\beta = \frac{m_3^p - m_2^p}{m_3^p - m_1^p}$$

By definition of generalized modified (p, h) convex stochastic process, we have

$$\begin{aligned} \psi(m_2, \cdot) &= \psi(\beta m_1^p + (1-\beta)m_3^p) \\ &\leq h(\beta)\psi(m_1, \cdot) + (1-h(\beta))\psi(m_3, \cdot) \\ &\leq h(\beta)\psi(m_1, \cdot) + (\psi(m_3, \cdot) - h(\beta)\psi(m_3, \cdot)) \\ &\leq (\psi(m_3, \cdot) + h(\beta)\psi(m_1, \cdot) - h(\beta)\psi(m_3, \cdot)) \\ &\leq (\psi(m_3, \cdot) + h(\beta)(\psi(m_1, \cdot) - \psi(m_3, \cdot))) \end{aligned}$$

$$\psi(m_2, \cdot) \leq (\psi(m_3, \cdot) + \eta h(\beta)(\psi(m_1, \cdot), \psi(m_3, \cdot)))$$

So,

$$0 \leq (\psi(m_3, \cdot) - \psi(m_2, \cdot) + \eta h(\beta)(\psi(m_1, \cdot), \psi(m_3, \cdot)))$$

$$0 \leq (\psi(m_3, \cdot) - \psi(m_2, \cdot) + \eta h \left(\frac{m_2^p - m_3^p}{m_1^p - m_3^p} \right) (\psi(m_1, \cdot), \psi(m_3, \cdot)))$$

$$0 \leq (\psi(m_3, \cdot) - \psi(m_2, \cdot) + \eta \frac{h(m_2^p - m_3^p)}{h(m_1^p - m_3^p)} (\psi(m_1, \cdot), \psi(m_3, \cdot)))$$

$$0 \leq (\psi(m_3, \cdot) - \psi(m_2, \cdot)(h(m_1^p - m_3^p)) + \eta h(m_2^p - m_3^p)(\psi(m_1, \cdot), \psi(m_3, \cdot)))$$

, Hence

$$\det \begin{vmatrix} h(m_3^p - m_2^p) & \psi(m_2, \cdot) - \psi(m_3, \cdot) \\ h(m_3^p - m_1^p) & \eta(\psi(m_1, \cdot), \psi(m_3, \cdot)) \end{vmatrix} \geq 0$$

.For the reverse inequality, take $m_1, m_2 \in I$ with $m_1 \leq m_2$. Choose any $\beta \in (0, 10)$, then we have

$$m_1^p \leq \beta m_1^p + (1 - \beta)m_2^p \leq m_2^p$$

so, the above determinant is;

$$0 \leq h(m_2^p - m_1^p)\eta(\psi(m_1, \cdot), \psi(m_2, \cdot)) - h(m_2^p - m_1^p)(\psi(m_1, \cdot) - \psi(m_2, \cdot))$$

can be written as,

$$\begin{aligned} 0 &\leq [h(m_2^p) - h[\beta m_1^p + (1 - \beta)m_2^p]]\eta(\psi(m_1, \cdot), \psi(m_2, \cdot)) - h(m_2^p - m_1^p)(\psi[\beta m_1^p + (1 - \beta)m_2^p]^{\frac{1}{p}}, \cdot) - \psi(m_2, \cdot)) \\ &0 \leq [h(m_2^p) - h(\beta)(m_1^p) - h(m_2^p) + h(\beta)m_2^p]\eta(\psi(m_1, \cdot), \psi(m_2, \cdot)) \\ &\quad - h(m_2^p - m_1^p)(\psi[\beta m_1^p + (1 - \beta)m_2^p]^{\frac{1}{p}}, \cdot) + h(m_2^p - m_1^p)\psi(m_2, \cdot). \end{aligned}$$

implies that,

$$\begin{aligned} &h(m_2^p - m_1^p)(\psi[\beta m_1^p + (1 - \beta)m_2^p]^{\frac{1}{p}}, \cdot) \\ &\leq h(\beta)(m_2^p - m_1^p)\eta(\psi(m_1, \cdot), \psi(m_2, \cdot)) + h(m_2^p - m_1^p)\psi(m_2, \cdot) \\ &(\psi[\beta m_1^p + (1 - \beta)m_2^p]^{\frac{1}{p}}, \cdot) \leq \psi(m_2, \cdot) + \beta\eta(\psi(m_1, \cdot), \psi(m_2, \cdot)). \end{aligned}$$

which is as required. □

4 Results for Jensen, Hermite-Hadamard and Ostrowski type Inequalities

4.1 Jensen type inequality

We will prove Jensen type inequality for modified (p,h)-convex stochastic process. let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be the modified (p,h)-convex stochastic process. For $l, m \in J$ and $\beta_1 + \beta_2 = 1$, we have,

$$\psi[(\beta_1 l^p + \beta_2 m^p)^{\frac{1}{p}}, \cdot] \leq h(\beta_1)(\psi(l, \cdot) + (1 - h(\beta_1))\psi(m, \cdot))$$

$$\leq h(\beta_1)(\psi(l, \cdot) + \psi(m, \cdot) - h(\beta_1)\psi(m, \cdot))$$

$$\leq \psi(m, \cdot) + h(\beta_1)(\psi(l, \cdot) - h(\beta_1)\psi(m, \cdot))$$

can be written as,

$$\leq \psi(m, \cdot) + h(\beta_1)(\psi(l, \cdot) - \psi(m, \cdot))$$

by η -convex stochastic process,

$$\leq \psi(m, \cdot) + h(\beta_1)\eta((\psi(l, \cdot), \psi(m, \cdot)))$$

so,

$$\psi[(\beta_1 x_1^p + \beta_2 m^p)^{\frac{1}{p}}, \cdot] \leq \psi(m, \cdot) + h(\beta_1)\eta((\psi(l, \cdot), \psi(m, \cdot)))$$

Also,

when, $k > 2$ for $l_1, l_2, l_3, \dots, l_k \in J$,

$$\sum_{r=1}^k \beta_r = 1,$$

and,

$$M_j = \sum_{s=1}^k \beta_j,$$

we have,

$$\begin{aligned} \psi \left[\left(\sum_{r=1}^k \beta_r l_r^p \right)^{\frac{1}{p}}, \cdot \right] &= \psi \left[M_{k-1} \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} + a_k l_n^p \right)^{\frac{1}{p}}, \cdot \right] \\ &\leq h(M_{k-1}) \left(\psi \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} \right)^{\frac{1}{p}}, \cdot \right) + (1 - h(M_{k-1}))(\psi_k, \cdot). \end{aligned}$$

we can write,

$$\leq h(M_{k-1}) \left(\psi \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} \right)^{\frac{1}{p}}, \cdot \right) + ((\psi_k, \cdot) - h(M_{k-1})(\psi_k, \cdot)).$$

finally,

$$\leq (\psi_k, \cdot) + h(M_{k-1}) \left[\left(\psi \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} \right)^{\frac{1}{p}}, \cdot \right) - (\psi_k, \cdot) \right].$$

$$\psi \left[\sum_{r=1}^k (\beta_1 l_1^p + \beta_2 l_2^p)^{\frac{1}{p}}, \cdot \right] \leq (\psi_k, \cdot) + h(M_{k-1}) \eta \left[\left(\psi \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} \right)^{\frac{1}{p}}, \cdot \right), (\psi_k, \cdot) \right]. \quad (4.1)$$

This is the jensen type inequality for generalized modified (p,h)-convex stochastic process.

Theorem 4.1. Consider $\psi : J \times \Pi \rightarrow \mathbb{R}$ be the modified (p,h)-convex stochastic process, and $\eta : X \times Y \rightarrow \mathbb{R}$ be the non-decreasing non-negatively sub-linear in first variable. If

$$M_r = \sum_{s=1}^r \beta_s$$

for $r = 1, 2, 3, \dots, k$ such that $M_k = 1$, then

$$\psi \left[\sum_{r=1}^k (\beta_1 l_1^p + \beta_2 l_2^p)^{\frac{1}{p}}, \cdot \right] \leq (\psi_k, \cdot) + \sum_{r=1}^{k-1} h(M_r) \eta \psi(l_r, l_r + 1, l_r + 2, \dots, l_k, \cdot). \quad (4.2)$$

$$\eta_\psi(l_r, l_{r+1}, l_{r+2}, \dots, l_k, \cdot) = \eta(\eta_\psi(l_r, l_{r+1}, l_{r+2}, \dots, l_{k-1}, \cdot))$$

and,

$$\eta_\psi(l, \cdot) = \psi(l, \cdot)$$

for all $l \in J$.

Proof. As we know that according to jensen-type inequality

$$\psi \left[\left(\sum_{r=1}^k \beta_r l_r^p + \beta_2 l_2^p \right)^{\frac{1}{p}}, \cdot \right] \leq h(M_{K-1}) \left(\psi \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} + a_k l_k^p \right)^{\frac{1}{p}}, \cdot \right) + (1 - h(M_{k-1}))(\psi_k, \cdot)$$

$$\leq h(M_{k-1}) \left(\psi \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} \right)^{\frac{1}{p}}, \cdot \right) + (1 - h(M_{k-1}))(\psi_k, \cdot)$$

$$\leq h(M_{K-1}) \left(\psi \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} \right)^{\frac{1}{p}}, \cdot \right) + ((\psi_k, \cdot) - h(M_{k-1})(\psi_k, \cdot))$$

implies that,

$$\leq (\psi_k, \cdot) + h(M_{k-1}) \left[\left(\psi \left(\sum_{r=1}^k \frac{\beta_r l_r^p}{M_{k-1}} \right)^{\frac{1}{p}}, \cdot \right) - (\psi_k, \cdot) \right]$$

further more,

$$\leq (\psi_k, \cdot) + h(M_{k-1}) \eta \left[\psi \left(\frac{M_{n-2}}{T_{k-1}} \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}} \right) + \frac{\beta_{k-1} l_{k-1}}{M_{k-2}} \right), \psi_k, \cdot \right]$$

$$\leq (\psi_k, \cdot) +$$

$$h(T_{k-1})\eta \left[h \left(\frac{M_{k-2}}{M_{k-1}} \right) \psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right) + \left(1 - h \left(\frac{T_{k-2}}{M_{k-1}} \right) \right) \psi(l_{k-1}, \cdot), \psi(l_k, \cdot) \right]$$

$$\leq (\psi_k, \cdot) +$$

$$h(M_{k-1})h \left(\frac{M_{k-2}}{M_{k-1}} \right) \eta \left[\psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right) + \psi(l_{k-1}, \cdot) - h \left(\frac{M_{k-2}}{M_{k-1}} \right) \psi(l_{k-1}, \cdot), \psi(l_n, \cdot) \right]$$

$$\leq (\psi_k, \cdot) + h(M_{k-1})\eta \left[\psi(l_{k-1}, \cdot) + h \left(\frac{M_{k-2}}{M_{k-1}} \right) \psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right) - \psi(l_{k-1}, \cdot), \psi(l_k, \cdot) \right]$$

now,

$$\leq (\psi_k, \cdot) + h(M_{k-1})\eta \left[\psi(l_{k-1}, \cdot) + h \left(\frac{M_{k-2}}{M_{k-1}} \right) \eta \left[\psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right), \psi(l_{k-1}, \cdot) \right], \psi(l_k, \cdot) \right]$$

similarly,

$$\leq (\psi_k, \cdot) + h(M_{k-1}) \left[\psi(l_{k-1}, \cdot) + h \left(\frac{M_{k-2}}{M_{k-1}} \right) \eta \left[\psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right), \psi(l_{k-1}, \cdot) \right] - \psi(l_k, \cdot) \right]$$

$$\leq (\psi_k, \cdot) + h(M_{k-1})\psi(l_{k-1}, \cdot) +$$

$$h(M_{k-2})\eta \left[\psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right), h(M_{k-1})\psi(l_{k-1}, \cdot) \right] - h(M_{k-1})\psi(l_n, \cdot)$$

$$\leq (\psi_k, \cdot) - h(M_{k-1})\psi(l_k, \cdot) + h(M_{k-1})\psi(l_{k-1}, \cdot) +$$

$$h(M_{k-2})\eta \left[\psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right), h(M_{k-1})\psi(l_{k-1}, \cdot) \right]$$

implies,

$$\leq (\psi_k, \cdot) + h(M_{k-1}) (\psi(l_k, \cdot) - \psi(l_{k-1}, \cdot)) +$$

$$h(M_{k-2})\eta \left[\psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right), \psi(l_{k-1}, \cdot) \right]$$

$$\leq (\psi_k, \cdot) + h(M_{k-1})\eta (\psi(l_k - 1, \cdot), \psi(l_k, \cdot)) +$$

$$\begin{aligned}
& h(M_{k-2})\eta \left[\psi \left(\sum_{r=1}^{k-2} \left(\frac{\beta_r l_r^p}{M_{k-2}} \right)^{\frac{1}{p}}, \cdot \right), \psi(l_{k-1}, \cdot) \right] \\
& \leq (\psi_k, \cdot) + h(M_{K-1})\eta(\psi(l_{k-1}, \cdot), \psi(l_k, \cdot)) + \\
& h(M_{k-2})\eta(\psi(l_{k-1}, \cdot), \psi(l_{k-1}, \cdot), \psi(l_k, \cdot)) + \dots + h(M_1)\eta(\psi(l_1, \cdot), \psi(l_2, \cdot), \dots, \psi(l_{k-1}, \cdot), \psi(l_k, \cdot)) \\
& \psi \left[\sum_{r=1}^k (\beta_1 l_1^p + \beta_2 l_2^p)^{\frac{1}{p}}, \cdot \right] \leq (\psi_k, \cdot) + \sum_{r=1}^{k-1} h(M_r)\eta\psi(l_r, l_r + 1, l_r + 2, \dots, l_n, \cdot).
\end{aligned}$$

□

4.2 Hermite-Hadamard type inequality

Now, we can establish the inequality for generalized modified (p,h)-convex stochastic process that is connected to the Hermite-Hadamard type Ineqyuality.

Theorem 4.2. *For any J that is subset of positive Real numbers($0 < J < 1$) and $p > 0$, let a measured generalised modified (p,h) convex stochastic process $\psi : [l, m] \times \Pi \rightarrow \mathbb{R}$ which is integerable. Then for any $l, m \in J, (l < m)$ there is the underlying inequity, which is always true.*

$$\begin{aligned}
& \psi \left(\left[\frac{l^p + m^p}{2} \right]^{\frac{1}{p}}, \cdot \right) \leq \frac{p}{l^p + m^p} \int_l^m z^{p-1} \psi(u, \cdot) dz. \\
& \leq \left(\frac{\psi(l, \cdot) + \psi(m, \cdot)}{2} \right) + \frac{1}{2} [\eta(\psi(l, \cdot), \psi(m, \cdot)) + \psi(m, \cdot), \psi(l, \cdot)] \int_0^1 h(q) dq \quad (4.3)
\end{aligned}$$

Proof. Let,

$$\delta_1^p = ql^p + (1 - q)m^p$$

,

$$\delta_2^p = (1 - q)l^p + qm^p$$

$$\psi \left(\left[\frac{l^p + m^p}{2} \right]^{\frac{1}{p}}, \cdot \right) = \psi \left(\left[\frac{\delta_1^p + \delta_2^p}{2} \right]^{\frac{1}{p}}, \cdot \right)$$

$$\psi \left(\left[\frac{l^p + m^p}{2} \right]^{\frac{1}{p}}, \cdot \right) = \psi \left(\left(\frac{1}{2} [ql^p + (1 - q)m^p]^{\frac{1}{p}}, \cdot \right)^p + \frac{1}{2} \left([(1 - q)l^p + qm^p]^{\frac{1}{p}}, \cdot \right)^p \right)$$

$$\leq \psi \left([(1 - q)l^p + qm^p]^{\frac{1}{p}}, \cdot \right) + h \left(\frac{1}{2} \right) \eta \left(\psi \left([ql^p + (1 - q)m^p]^{\frac{1}{p}}, \cdot \right), \psi \left([(1 - q)l^p + qm^p]^{\frac{1}{p}}, \cdot \right) \right)$$

integrate according to "q" on given interval

$$\begin{aligned} & \psi \left(\left[\frac{l^p + m^p}{2} \right]^{\frac{1}{p}}, \cdot \right) \\ & \leq \int_0^1 \psi \left([(1-q)l^p + qm^p]^{\frac{1}{p}}, \cdot \right) dq \\ & + h \left(\frac{1}{2} \right) \int_0^1 \eta \left(\psi \left([ql^p + (1-q)m^p]^{\frac{1}{p}}, \cdot \right), \psi \left([(1-q)l^p + qm^p]^{\frac{1}{p}}, \cdot \right) \right) dq \end{aligned}$$

Let,

$$\begin{aligned} z^p &= ql^p + (1-q)m^p \\ pz^{p-1}dz &= (l^p - m^p)dq \\ \frac{p}{(l^p - m^p)} z^{p-1}dz &= dq \end{aligned}$$

$$\begin{aligned} & \leq \left[1 - h\left(\frac{1}{2}\right) \right] \frac{p}{(m^p - l^p)} \int_l^m z^{p-1} \psi(z, \cdot) dz + h\left(\frac{1}{2}\right) \frac{p}{(l^p - m^p)} \int_m^l z^{p-1} \psi(z, \cdot) dz \\ & \leq \left[1 - h\left(\frac{1}{2}\right) \right] \frac{p}{(m^p - l^p)} \int_l^m z^{p-1} \psi(z, \cdot) dz + h\left(\frac{1}{2}\right) \frac{p}{(m^p - l^p)} \int_l^m z^{p-1} \psi(z, \cdot) dz \\ & \leq \frac{p}{(m^p - l^p)} \int_l^m z^{p-1} \psi(z, \cdot) dz \\ & + h\left(\frac{1}{2}\right) \frac{p}{(m^p - l^p)} \int_l^m z^{p-1} \psi(z, \cdot) dz - h\left(\frac{1}{2}\right) \frac{p}{(m^p - l^p)} \int_l^m z^{p-1} \psi(z, \cdot) dz \end{aligned}$$

implies that,

$$\psi \left(\left[\frac{l^p + m^p}{2} \right]^{\frac{1}{p}}, \cdot \right) \leq \frac{p}{(m^p - l^p)} \int_l^m z^{p-1} \psi(z, \cdot) dz$$

now solve for this,

$$\int_l^m z^{p-1} \psi(z, \cdot) dz$$

we can write this as,

$$\int_l^m z^{p-1} \psi(z, \cdot) dz = \frac{m^p - l^p}{p} \times \frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz$$

As we know that,

$$\frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz = \int_0^1 \psi \left([ql^p + (1-q)m^p]^{\frac{1}{p}}, \cdot \right) dq$$

so above equation becomes,

$$\int_l^m z^{p-1} \psi(z, \cdot) dz = \frac{m^p - l^p}{p} \int_0^1 \psi \left([ql^p + (1-q)m^p]^{\frac{1}{p}}, \cdot \right) dq$$

According to the generalised modified (p,h) convex stochastic process,

$$\int_l^m z^{p-1} \psi(z, \cdot) dz = \frac{m^p - l^p}{p} \int_0^1 (h(q)\psi(l, \cdot) + \psi(m, \cdot) - h(q)\psi(m, \cdot)) dq$$

so,

$$\frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz \leq \int_0^1 (h(q)\psi(l, \cdot) + \psi(m, \cdot) - h(q)\psi(m, \cdot)) dq$$

implies that,

$$\begin{aligned} \frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz &\leq \left(\psi(m, \cdot) + \int_0^1 (h(q)\psi(l, \cdot) - h(q)\psi(m, \cdot)) dq \right) . \\ \frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz &\leq \left(\psi(m, \cdot) + \int_0^1 (h(q) (\psi(l, \cdot) - \psi(m, \cdot))) dq \right) \end{aligned} \quad (4.4)$$

Finally it becomes,

$$\frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz \leq \left(\psi(m, \cdot) + \int_0^1 \eta (h(q) (\psi(l, \cdot), \psi(m, \cdot))) dq \right)$$

Again from the above equation,

$$\int_l^m z^{p-1} \psi(z, \cdot) dz = \frac{m^p - l^p}{p} \times \frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot)$$

we know that,

$$\frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) = \int_0^1 \psi ([(1-q)l^p + qm^p], \cdot)$$

implies that

$$\int_l^m z^{p-1} \psi(z, \cdot) dz = \frac{m^p - l^p}{p} \int_0^1 \psi(l, \cdot) - h(q)\psi(l, \cdot) + h(q)\psi(m, \cdot) dq$$

$$\begin{aligned}
\int_l^m z^{p-1} \psi(z, \cdot) dz &\leq \frac{m^p - l^p}{p} \int_0^1 h(q) \psi(m, \cdot) - h(q) \psi(l, \cdot) + \psi(l, \cdot) d\theta \\
&\leq \frac{m^p - l^p}{p} \left(\psi(l, \cdot) + \int_0^1 h(q) (\psi(m, \cdot) - \psi(l, \cdot)) dq \right) \\
&\leq \frac{m^p - l^p}{p} \left(\psi(l, \cdot) + \int_0^1 \eta h(q) (\psi(m, \cdot), \psi(l, \cdot)) dq \right) \\
\frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz &\leq \left(\psi(l, \cdot) + \int_0^1 \eta h(q) (\psi(m, \cdot), \psi(l, \cdot)) dq \right). \tag{4.5}
\end{aligned}$$

adding the equation (3.2) and (3.3)

$$\begin{aligned}
&\frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz + \frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz \\
&\leq \left(\psi(l, \cdot) + \int_0^1 \eta h(q) (\psi(m, \cdot), \psi(l, \cdot)) dq + (\psi(m, \cdot)) \right. \\
&\quad \left. + \left(\int_0^1 \eta (h(q) (\psi(l, \cdot), \psi(m, \cdot))) dq \right) \right) \\
&\quad \frac{2p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz \\
&\quad \leq (\psi(l, \cdot) + \psi(m, \cdot)) \\
&\quad + \int_0^1 \eta h(q) (\psi(m, \cdot), \psi(l, \cdot)) dq + \int_0^1 \eta h(q) (\psi(l, \cdot), \psi(m, \cdot)) dq \\
&\leq \left(\frac{\psi(l, \cdot) + \psi(m, \cdot)}{2} \right) + \frac{1}{2} [\eta (\psi(l, \cdot), \psi(m, \cdot)) + \eta (\psi(m, \cdot), \psi(l, \cdot))] \int_0^1 h(q) dq
\end{aligned}$$

so,

$$\frac{2p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz$$

$$\leq \left(\frac{\psi(l, \cdot) + \psi(m, \cdot)}{2} \right) + \frac{1}{2} [\eta(\psi(l, \cdot), \psi(m, \cdot)) + \eta(\psi(m, \cdot), \psi(l, \cdot))] \int_0^1 h(q) dq.$$

□

The result is proved now.

4.3 Ostrowski type inequality

The following Lemma is necessary to prove this inequality according to the selected definition (generalized modified (p,h)-convex stochastic process)

Lemma 4.1. *Let $\psi : J \times \Pi \rightarrow \mathbb{R}$ is the generalised modified (p,h) convex stochastic process that is mean square differentiable in J° . If ψ' is mean square integerable in the interval $[b_1, b_2]$, in which $b_1, b_2 \in I$ and $b_1 < b_2$ with $p \in \mathbb{R}$, then the following inequality holds.*

$$\begin{aligned} & \psi(y, \cdot) - \frac{p}{b_2^p - b_1^p} \int_{b_1}^{b_2} \frac{\psi(v, \cdot)}{v^{1-p}} dv \\ & \leq \frac{1}{p(b_2^p - b_1^p)} \left[(y^p - b_1^p)^2 \int_0^1 \frac{s}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} \psi' \left((sy^p + (1-s)b_1^p)^{\frac{1}{p}}, \cdot \right) ds \right] \\ & \quad - \frac{1}{p(b_2^p - b_1^p)} \left[(b_2^p - y^p)^2 \int_0^1 \frac{s}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} \psi' \left((sy^p + (1-s)b_2^p)^{\frac{1}{p}}, \cdot \right) ds \right]. \end{aligned}$$

Theorem 4.3. *Consider the mapping, $\psi : J \subseteq (0, \infty) \rightarrow \mathbb{R}$ be differentiable function on J° and $b_1, b_2 \in J^\circ$ with $b_1 < b_2$ and $p \in \mathbb{R}/0$ and $\psi' \in L[b_1, b_2]$. If $|\psi'|^k$ is generalized modified (p,h)convex stochastic process in $[b_1, b_2]$ for $k \geq 1$, then for all $y \in [b_1, b_2]$ we get,*

$$\begin{aligned} & \left| \psi(y, \cdot) - \frac{p}{b_2^p - b_1^p} \int_{b_1}^{b_2} \frac{\psi(v, \cdot)}{v^{1-p}} dv \right| \\ & \leq \frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k |\psi'(b_1, \cdot)|^k + h(s)\eta [(|\psi'(y, \cdot)|^k, |\psi'(b_1, \cdot)|^k)]}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}} \\ & \quad - \frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k |\psi'(b_2, \cdot)|^k + h(s)\eta [(|\psi'(y, \cdot)|^k, |\psi'(b_2, \cdot)|^k)]}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}} \end{aligned}$$

Proof. From Lemma we can construct the proof,

$$\begin{aligned} & \left| \psi(y, \cdot) - \frac{p}{b_2^p - b_1^p} \int_{b_1}^{b_2} \frac{\psi(v, \cdot)}{v^{1-p}} dv \right| \\ & \leq \frac{1}{p(b_2^p - b_1^p)} \left[(y^p - b_1^p)^2 \int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} s \psi' \left((sy^p + (1-s)b_1^p)^{\frac{1}{p}}, \cdot \right) ds \right] \\ & \quad - \frac{1}{p(b_2^p - b_1^p)} \left[(b_2^p - y^p)^2 \int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} s \psi' \left((sy^p + (1-s)b_2^p)^{\frac{1}{p}}, \cdot \right) ds \right] \end{aligned}$$

multiply

$$\left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}}$$

and,

$$\left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{-1}{k}}$$

in first term. And multiply

$$\left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}}$$

and

$$\left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{-1}{k}}$$

in second term we get,

$$\begin{aligned} & \left| \psi(y, \cdot) - \frac{p}{b_2^p - b_1^p} \int_{b_1}^{b_2} \frac{\psi(v, \cdot)}{v^{1-p}} dv \right| \\ & \leq \frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{-1}{k}} \times \int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \\ & \quad \times \int_0^1 s\psi' \left((sy^p + (1-s)b_1^p)^{\frac{1}{p}}, \cdot \right) ds \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}} \\ & \quad - \frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)} \int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{-1}{k}} \\ & \quad \int_0^1 s\psi' \left((sy^p + (1-s)b_2^p)^{\frac{1}{p}}, \cdot \right) ds \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}} \end{aligned}$$

implies that,

$$\begin{aligned} & \left| \psi(y, \cdot) - \frac{p}{b_2^p - b_1^p} \int_{b_1}^{b_2} \frac{\psi(v, \cdot)}{v^{1-p}} dv \right| \\ & \leq \frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \\ & \quad \times \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}} \int_0^1 s\psi' \left((sy^p + (1-s)b_1^p)^{\frac{1}{p}}, \cdot \right) ds \end{aligned}$$

$$\begin{aligned} & -\frac{(b_2^p - y^p)^2}{p(b_2^P - b_1^P)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \\ & \times \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}} \int_0^1 s\psi' \left((sy^p + (1-s)b_2^p)^{\frac{1}{p}}, \cdot \right) ds \end{aligned}$$

now in next step,

$$\begin{aligned} & \left| \psi(y, \cdot) - \frac{p}{b_2^p - b_1^p} \int_{b_1}^{b_2} \frac{\psi(v, \cdot)}{v^{1-p}} dv \right| \\ & \leq \frac{(y^p - b_1^p)^2}{p(b_2^P - b_1^P)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \\ & \times \left(\int_0^1 \frac{s^k}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} \left| \psi' \left((sy^p + (1-s)b_1^p)^{\frac{1}{p}}, \cdot \right) \right|^k ds \right)^{\frac{1}{k}} \\ & -\frac{(b_2^p - y^p)^2}{p(b_2^P - b_1^P)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \\ & \times \left(\int_0^1 \frac{s^k}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} \left| \psi' \left((sy^p + (1-s)b_2^p)^{\frac{1}{p}}, \cdot \right) \right|^k ds \right)^{\frac{1}{k}} \\ & \leq \frac{(y^p - b_1^p)^2}{p(b_2^P - b_1^P)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \\ & \times \left(\int_0^1 \frac{s^k}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} h(s) |\psi'(y, \cdot)|^k + (1-h(s)) |\psi'(b_1, \cdot)|^k ds \right)^{\frac{1}{k}} \\ & -\frac{(b_2^p - y^p)^2}{p(b_2^P - b_1^P)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \\ & \times \left(\int_0^1 \frac{s^k}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} h(s) |\psi'(y, \cdot)|^k + (1-h(s)) |\psi'(b_2, \cdot)|^k ds \right)^{\frac{1}{k}} \\ & \leq \frac{(y^p - b_1^p)^2}{p(b_2^P - b_1^P)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k [h(s) (|\psi'(y, \cdot)|^k + (1-h(s)) |\psi'(b_1, \cdot)|^k)]}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}} \\ & -\frac{(b_2^p - y^p)^2}{p(b_2^P - b_1^P)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k [h(s) |\psi'(y, \cdot)|^k + (1-h(s)) |\psi'(b_2, \cdot)|^k]}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}} \end{aligned}$$

by the definition of η stochastic process,

$$\leq \frac{(y^p - b_1^p)^2}{p(b_2^P - b_1^P)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k [h(s) (|\psi'(y, \cdot)|^k + |\psi'(b_1, \cdot)|^k - h(s) |\psi'(b_1, \cdot)|^k)]}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}}$$

$$-\frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k [h(s)|\psi'(y, \cdot)|^k + |\psi'(b_2, \cdot)|^k - h(s)|\psi'(b_2, \cdot)|^k]}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}}$$

implies that,

$$\leq \frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k [|\psi'(b_1, \cdot)|^k + h(s)[(|\psi'(y, \cdot)|^k - |\psi'(b_1, \cdot)|^k)]]}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}}$$

$$-\frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k [|\psi'(b_2, \cdot)|^k + h(s)[|\psi'(y, \cdot)|^k - |\psi'(b_2, \cdot)|^k]}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}}$$

finally,

$$\leq \frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k |\psi'(b_1, \cdot)|^k + s^k (h(s)\eta [(|\psi'(y, \cdot)|^k, |\psi'(b_1, \cdot)|^k)])}{(sy^p + (1-s)b_1^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}}$$

$$-\frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{1-\frac{1}{k}} \times \left(\int_0^1 \frac{s^k |\psi'(b_2, \cdot)|^k + s^k (h(s)\eta [(|\psi'(y, \cdot)|^k, |\psi'(b_2, \cdot)|^k)])}{(sy^p + (1-s)b_2^p)^{1-\frac{1}{p}}} ds \right)^{\frac{1}{k}}.$$

□

5 Conclusion

Statistics has a variety of applications for stochastic processes, which leads to a lot of other disciplines, for example, Kolmogorov–Smirnov test to the distributional equality[8][3][6]. Sequential analysis [25][1] (a rigorous technique to dynamically end an A/B test is one of the other uses. (Stopping rules are obtained by equating discontinuous situations using with continuous and discrete equivalents, where the appropriate statistic procedure proceeds a random equations) and the most rapid detection[5][7]. The decision rules are based on the attributes of random process striking times once more. Meanwhile, in applied sciences and mathematics, convexity has an essential rule. We introduce Generalized modified (p,h)-convex stochastic functions in this article, which combine p-convexity with modified h-convexity. The Hermite-Hadamard, Jensen's, and Ostrowski type inequalities were derived from the fundamental characteristics of Generalized modified (p,h)-convex stochastic process. we can further conclude Fejer, schur type inequalities for our undergoing problem.

6 Bibliography

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