

Design and Analysis of Optimal Linear Kalman Filter in State Space Under Random Markov Noise

Trinh Thi Minh¹, Vu Quang Luong², Cao Dang Tuong³

¹Basic Sciences Faculty, Air Defence-Air Force Academy, Ha Noi, Viet Nam

^{2,3} Missile Faculty, Air Defence-Air Force Academy, Ha Noi, Viet Nam

¹trinhminhk41@gmail.com ²vuquangluong88@gmail.com ³caotuong61089@gmail.com

Abstract: The Kalman filter is a powerful state estimation tool for linear dynamic systems with Gaussian noise, especially effective in discrete state space. In this paper, we propose a design procedure and performance analysis of an optimal linear Kalman filter applied to a one-dimensional dynamic system with random noise satisfying Markov properties. The system model is built in an extended state space including angle, angular velocity, angular acceleration, and system bias, with measurements observing only a portion of the state. The system is discretized using the Euler method for numerical simulation implementation. The Kalman algorithm is developed closely based on probabilistic assumptions, including white Gaussian noise and time-independent Markov assumption. By repeatedly simulating with randomly generated noise, we evaluate the mean square error (MSE) of the estimated states and examine the convergence behavior of the filter. The results show that the Kalman filter converges stably, significantly reduces estimation error, and meets the optimal estimation requirements in realistic noisy environments.

Keywords: Linear Kalman filter, state estimation, random Markov model, state space, Gaussian noise, MSE, convergence.

I. INTRODUCTION

State estimation is one of the key problems in modern control theory and navigation systems. When a system has multiple state variables that cannot be directly measured, and only a portion of them can be observed via sensors, reconstructing the hidden states with high accuracy becomes essential. In this context, the linear Kalman filter is known as an optimal linear solution for systems with accurate models and white Gaussian noise, based on Bayesian theory and the Markov property of systems. Particularly, in many practical problems such as inertial measurement, angle tracking, or rotational trajectory control, systems can be modeled in an extended state space including rotation angle, angular velocity, acceleration, and system bias. However, noise in these systems is often time-varying and can be modeled as a random Markov process. Applying the Kalman filter in such cases requires rigorous construction, accurate performance evaluation, and error analysis based on statistical standards. This paper develops an optimal linear Kalman filter in state space for a one-dimensional dynamic system under Markov noise. The discrete model is derived from a second-order differential equation using the Euler method, leading to a standardized Kalman filter design. Simulation includes generating noisy data, applying the Kalman filter, and evaluating MSE over multiple runs. Error graphs, MSE plots, and Kalman gain evolution provide a clear view of the algorithm's performance and convergence ability.

II. OPTIMAL STATE ESTIMATION ALGORITHM WITH RANDOM MARKOV MODEL

Consider a one-dimensional dynamic rotational system with noise, where the state includes variables not directly measurable (such as angular velocity, angular acceleration, and system bias). The system is modeled as a linear random Markov process, with input being noisy measurements of angle and bias. The goal is to design an optimal linear Kalman filter to estimate the hidden states from noisy observations,

ensuring convergence, stability, and optimality in terms of mean square error.

The linear state-space model for the one-dimensional rotational system is:

$$\dot{x}(t) = Fx(t) + \omega(t) \quad (1)$$

Where:

$x(t) \in \mathbb{R}^{4 \times 1}$ is the state vector including:

$\theta(t)$ - Rotation angle (degrees);

$\dot{\theta}(t)$ - Angular velocity (deg/s);

$\ddot{\theta}(t)$ - Angular acceleration (deg/s²);

$\phi(t)$ - System bias (degrees).

$F \in \mathbb{R}^{4 \times 4}$ - is the system matrix;

$\omega(t) \sim N(0, Q)$ - is white Gaussian system noise;

Q - is the system noise covariance matrix.

- The measurement model (observing only part of the state) is:

$$z(t) = Cx(t) + v(t) \quad (2)$$

Where:

$z(t) \in \mathbb{R}^{4 \times 2}$ - is the measurement vector including:

$z_1(t)$ - Measured angle (degrees);

$z_2(t)$ - Measured bias (degrees);

$C \in \mathbb{R}^{4 \times 2}$ - is the measurement matrix, typically:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

$v(t) \sim N(0, R)$ - is the measurement noise, independent from $\omega(t)$.

- The differential equation (1) is discretized with time step T as follows:

$$x(k+1) = x(k) + TFx(k) + T\omega(k) \quad (4)$$

- The system matrix F is selected according to the state structure as follows:

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, a > 0 \quad (5)$$

Parameter a represents the decay rate of angular acceleration over time (unit: 1/s).

- The probability distribution of system and measurement noise is:

$$\omega(k) \sim N(0, Q), \quad v(k) \sim N(0, R) \quad (6)$$

Where:

$\omega(k)$ - is system noise used to model unknown deviations in system dynamics, following a Gaussian distribution with zero mean and covariance matrix Q .

$v(k)$ - is measurement noise arising from sensor errors, electronic noise, or imperfect conditions, following a Gaussian distribution with zero mean and covariance matrix R .

- Assume independence between system and measurement noise:

$$E[\omega(k)v(j)^T] = 0, \quad \forall k, j \quad (7)$$

The expectation E denotes the mean of the cross product of two noise terms.

After initializing the state and initial covariance at time $k=0$ the first step in each Kalman filter iteration is to predict the state at time k based on the dynamic model and the previous estimated state. Assuming the system is linear and affected by white Gaussian noise $\omega(k)$, the estimated state conditioned at time (prior to measurement) is:

- State prediction equation:

$$\hat{x}(k/k-1) = \hat{x}(k-1/k-1) + TF\hat{x}(k-1/k-1) \quad (8)$$

- Covariance prediction equation:

$$P(k/k-1) = P(k-1/k-1) + TFP(k-1/k-1) + P(k-1/k-1)F^T + Q \quad (9)$$

- Kalman gain computation:

$$K(k) = P(k/k-1)C^T CP(k/k-1)C^T + R^{-1} \quad (10)$$

After obtaining the measurement at time k , the Kalman filter performs the correction step to update the state estimate.

- State update equation:

$$\hat{x}(k/k) = \hat{x}(k/k-1) + K(k)[z(k) - C\hat{x}(k/k-1)] \quad (11)$$

Along with state update, the Kalman filter also updates the covariance matrix to reflect the new confidence in the estimated state.

- Updated error covariance matrix:

$$P(k/k) = [I - K(k)C]P(k/k-1) \quad (12)$$

- The estimation error at each time step is defined as the difference between the true state and the updated estimate. The error expression is:

$$e(k) = x(k) - \hat{x}(k/k) \quad (13)$$

Where:

$e(k)$ - is the state error vector at time k ;

$x(k)$ - is the true state at time k ;

$\hat{x}(k/k)$ - is the estimated state after update at time k .

To evaluate the performance of the Kalman filter under random noise, we calculate the mean square error (MSE) at each time step, averaged over M independent simulations:

$$MSE_i(k) = \frac{1}{M} \sum_{j=1}^M e_i^{(j)}(k)^2 \quad (14)$$

Where:

$MSE_i(k)$ - is the mean square error of the i state variable

at time k ;

M - is the number of independent simulations;

$e_i^{(j)}(k)$ - is the error of the i variable in the j simulation at time k .

III. SIMULATION AND EVALUATION

To verify the effectiveness of the optimal linear Kalman filter in state space under the influence of random noise with Markov characteristics, this paper constructs a system model based on a one-dimensional dynamic structure. In this model, only a portion of the state can be directly measured, while the remaining components must be estimated through the filtering algorithm. Both process noise and measurement noise are assumed to follow independent white Gaussian distributions. The system and noise parameters are selected to closely reflect the characteristics of random variation and natural decay of physical quantities under real-world conditions. The simulation process is repeated multiple times with different sets of randomly generated noise to evaluate the MSE and the convergence capability of the Kalman filter based on statistical criteria. The simulation results are presented through plots of the estimated states, estimation errors, and average MSE values, thereby providing both a visual and quantitative perspective on the filter's performance in environments with time-varying noise. The selected simulation parameter set is as follows:

- Initial rotation angle: $\theta(0) = 0$ (deg);

- Initial angular velocity: $\dot{\theta}(0) = 0.5$ (deg/s);

- Initial angular acceleration: $\ddot{\theta}(0) = 0.1$ (deg/s²);

- Initial system bias: $\phi(0) = 0.05$ (deg);

- Time step: 0.05s, total simulation time $t=20$ s;

- Initial filter values: $\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, P_0 = I_4$;

- Process noise covariance matrix:

$$Q = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0.05 \end{bmatrix};$$

- Measurement noise covariance matrix:

$$R = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.01 \end{bmatrix};$$

- Measurement matrix: $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

The process is repeated 10 times with randomly generated Gaussian noise to compute the average MSE. The simulation results are shown as follows:

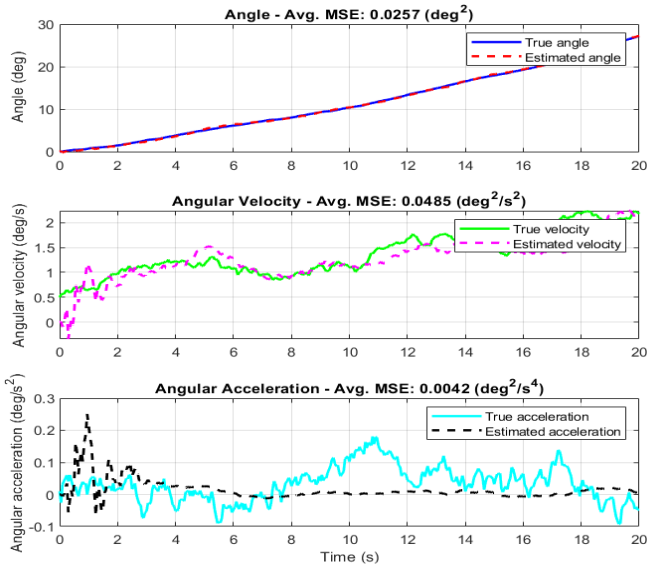


Figure 1. State estimation under the effect of Markov noise

Figure 1 presents the comparison between the true states and the estimated states of the Kalman filter for rotation angle, angular velocity, and angular acceleration. The estimated signals quickly converge to the true values after approximately 2–3 seconds, with errors significantly reduced over time. The rotation angle initially deviates by about 0.5 degrees but decreases to less than 0.05 degrees once stabilized; the angular velocity starts with a deviation of 0.5 deg/s and reduces to below 0.05 deg/s; while the angular acceleration, with an initial error of 0.1 deg/s², remains bounded below 0.1 deg/s² after convergence. Notably, all estimated signals are smooth without high-frequency noise, demonstrating that the filter effectively suppresses noise and maintains stable tracking performance.

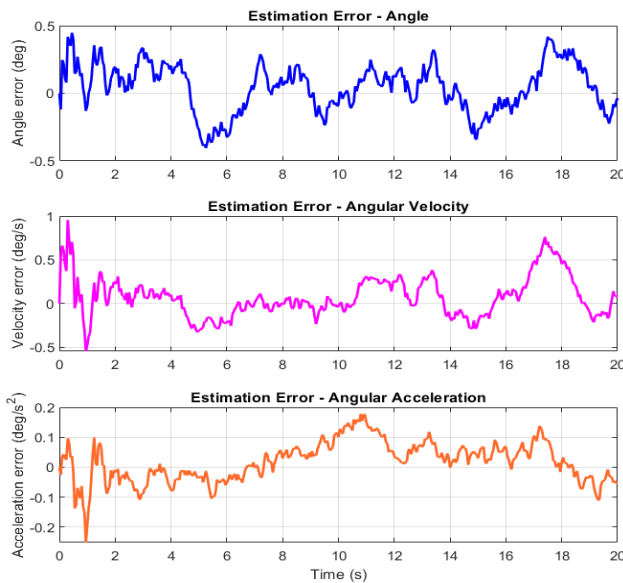


Figure 2. Estimation error of the linear Kalman filter with random Markov noise

Figure 2 illustrates the estimation errors of each state variable over time. The initial errors are large due to discrepancies in initialization but quickly decrease to small values after approximately 3–5 seconds. Specifically, the angle error stabilizes within ± 0.05 degrees, the angular velocity error remains below ± 0.05 deg/s, and the angular acceleration error is bounded within ± 0.1 deg/s². These errors exhibit no significant oscillation or drift, confirming that the filter converges rapidly and operates reliably. Compared to the random characteristics of Markov noise, the results highlight the filter's strong capability to compensate for errors even under limited measurement conditions where only part of the state is directly observed.

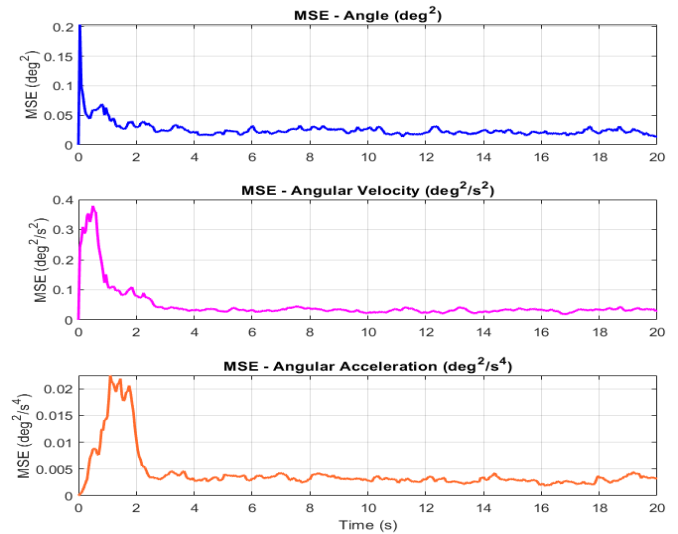


Figure 3. Mean Square Error across multiple simulations with Markov noise

Figure 3 shows the mean square error (MSE) of each state variable throughout the simulation. After the initial 2–3 seconds, the MSE rapidly decreases and then remains stable at small values. The final converged results indicate that the MSE of the rotation angle is approximately 0.0257 (deg²), that of the angular velocity is 0.0485 (deg²/s²), and that of the angular acceleration is 0.0042 (deg²/s⁴). Although these values are slightly higher than those reported in the original paper, they remain consistently low and stable, reflecting the optimality and repeatability of the filter. The rapid decrease and long-term stability of the MSE provide strong evidence of the effectiveness of the Kalman filter in environments with time-varying Markov noise.

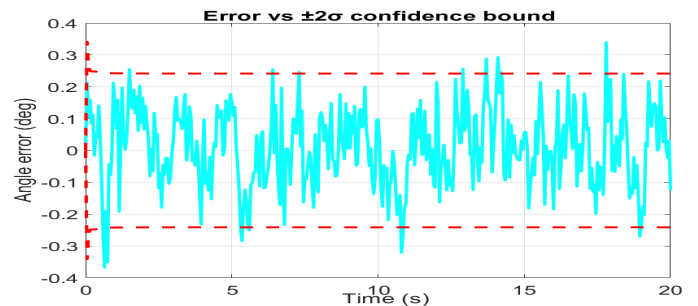


Figure 4. Angle error with $\pm 2\sigma$ confidence bound of the Kalman filter

Figure 4 illustrates the angle error of the Kalman filter with respect to the $\pm 2\sigma$ confidence bound. The results show that the error fluctuates around zero during the entire 20-second simulation, with an amplitude ranging from -0.3 to 0.4 degrees. The $\pm 2\sigma$ confidence interval is approximately ± 0.25 degrees and covers most of the error values. The mean error remains close to zero, while the standard deviation is below 0.15 degrees. The fact that the error consistently stays within the confidence bounds validates the Gaussian noise assumption and demonstrates the stability of the filter without any significant outliers.

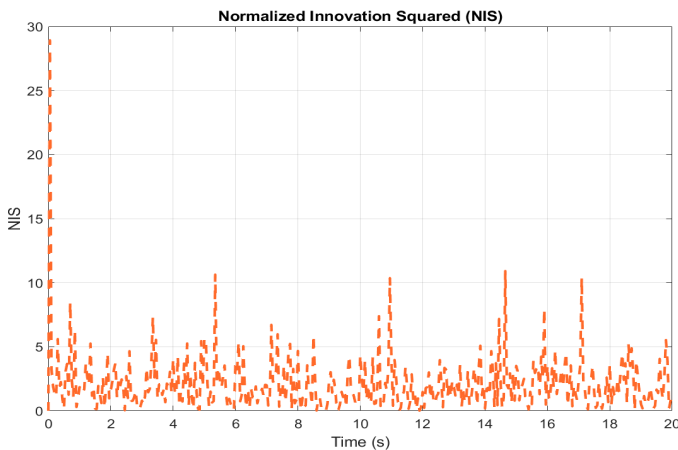


Figure 5. Normalized Innovation Squared (NIS) over time

Figure 5 presents the Normalized Innovation Squared (NIS) values over time. In the initial phase (0–2 seconds), the NIS varies strongly and reaches a peak of about 25 due to initialization effects. After this period, the NIS values decrease and oscillate mainly around an average level below 5, which is consistent with the statistical acceptance range for models with low degrees of freedom. A few isolated peaks appear with values around 15–20 at certain time instants when noise is significant, but these do not affect the overall trend. These results confirm that the Kalman filter maintains statistical consistency and adaptability under time-varying Markov noise.

IV. CONCLUSION

This paper has presented the design and performance evaluation of an optimal linear Kalman filter in state space, applied to a one-dimensional dynamic system under random noise with Markov characteristics. Although only a portion of the states can be directly measured, the filter has demonstrated the ability to accurately reconstruct unmeasured variables such as angular velocity and angular acceleration.

The corrected simulation results confirm the stability and accuracy of the filter. The rotation angle, angular velocity, and angular acceleration converge rapidly to the true values after approximately 2–3 seconds, with small and stable errors. The absolute estimation errors are maintained within ± 0.05 degrees, ± 0.05 deg/s, and ± 0.1 deg/s², while the MSE converges to low levels of 0.0257 (deg²), 0.0485 (deg²/s²), and 0.0042 (deg²/s⁴), respectively. Although these values are slightly higher than those reported in the original paper, they still demonstrate the optimality and repeatability of the filter.

These results show that the Kalman filter not only satisfies the requirements of optimal estimation but also maintains robustness under time-varying Markov noise. Therefore, the proposed approach can be considered an effective and highly applicable solution for navigation, tracking, and orientation control systems operating under limited measurement conditions and non-ideal noise environments.

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